3D XY scaling theory of the superconducting phase transition

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The intermediate 3D XY scaling theory of superconductivity at zero and nonzero magnetic fields is developed, based only upon the dimensional hypothesis $B \sim (\text{Length})^{-2}$. Universal as well as nonuniversal aspects of the theory are identified, including background terms and demagnetization effects. Two scaling regions are predicted: an "inner" region (very near the zero field superconducting transition, T_c), where the fields B, H, and H_{ex} differ substantially, due to the presence of diamagnetic fluctuations, and an "outer" region (away from T_c), where the fields can all be treated similarly. The characteristic field (H_0) and temperature (t_1) scales, separating the two regimes, are estimated. Scaling theories of the phase transition line, magnetization, specific heat, and conductivity are discussed. Multicritical behavior, involving critical glass fluctuations, is investigated along the transition line, $T_m(B)$, at nonzero fields.

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I. INTRODUCTION

The phase transition into the superconducting state in the high-T_c oxide materials has been the focus of much theoretical and experimental research. This transition is particularly interesting because mean field theory does not provide an adequate description; for both zero and nonzero magnetic fields, fluctuations play an important role. The challenge is to acquire a sound understanding of these fluctuations in different regimes of interest.

The superconducting transition corresponds to the melting of a vortex solid, and occurs along the line $T_m(B)$ in the field-temperature plane. When $B > 0$, the transition can be either first order, as seems to be the case for pure systems [1–3], or continuous, when systems contain disorder [4]. However, we must distinguish between the transitions at zero and nonzero fields. For example, although $T_m(B > 0)$ may be first order, the zero field transition, T_c, is expected to be continuous. The zero and finite field transitions must therefore belong to different universality classes, as corroborated experimentally [2,3]. If the transition $T_m(B > 0)$ is also continuous, it must be associated with a second type of critical behavior. At low fields, it is possible to observe both critical behaviors simultaneously [5–8]. The fact that the $B > 0$ transition line, with its distinct critical behavior, connects smoothly to the zero field transition [$T_c = \lim_{B\to 0} T_m(B)$] implies that $(T = T_c, B = 0)$ is a multicritical point [9].

As happens in two dimensional superconductors, the three dimensional (3D) superconducting phase transition occurs somewhat below the temperature where fluctuations of the order parameter *amplitude* drive the density of fluctuating Cooper pairs to zero. A growing body of experimental evidence [2,3,5–8,10–17] supports the idea that the zero field transition of strongly type-II superconductors, such as the high-T_c oxides, is in the "intermediate" 3D XY model class, which includes the λ-transition in ^{4}He. The essential fluctuations of this class involve the order parameter *phase*, but do not involve its amplitude or the vector potential.

When vector potential fluctuations are also considered, in addition to phase fluctuations of the order parameter, the ultimate critical behavior of the superconducting phase transition may belong to the "inverted" 3D XY model class [18]. The relative stiffness of the vector potential in strongly type-II superconductors, compared to the order parameter phase, causes this inverted critical regime to be very small. In this work, we focus our attention on the intermediate scaling region.

Critical fluctuations along the transition line $T_m(B > 0)$ are thought to be of the glass type, in the case of strong disorder [4,19,20]. However, experiments show that 3D XY fluctuations continue to be relevant for the scaling when $B > 0$. These include measurements of the specific heat [3,5,10,11,16], magnetization [2,5,11–13], penetration depth [14], and conductivity [5–8,16,17]. While 3D XY and glass fluctuations may coexist at low fields near the superconducting transition, they correspond to two distinct universality classes, with distinct exponents and scaling functions. This multicritical coexistence of superconducting fluctuations has been studied experimentally [5–8].

In this paper we consider intermediate 3D XY fluctuations, for low magnetic fields, $B \geq 0$ [21]. For simplicity we consider only the case of a continuous transition $T_m(B)$ at nonzero fields; details of the case of first order melting are presented elsewhere [22]. Some other theoretical treatments of the 3D XY model of superconductivity include Refs. [16,19,23–26], while some relevant numerical simulations are given in Ref. [27]. Although exact solutions to this problem are not yet in sight, a fruitful advance can be made using the scaling approach. In this method, a scaling form for the free energy is hypothesized by means of a dimensional analysis. Using this

ansatz, the theory is developed in a very general form, relying as little as possible on particular models for the superconductor. In addition to the free energy, we discuss the magnetization, specific heat, and conductivity. One important aspect of the present work is to clearly specify the nonuniversal parameters which enter the theory. These are of three types. One type comes from the smooth background terms. The second type is associated with demagnetization effects, and is related to sample geometry. The third type involves material-dependent constants which enter the scaling term of the free energy.

We also carefully distinguish between the different fields: the spatially averaged magnetic field B, its conjugate field H, and the external field $H_{\rm ex}$. The usual assumption that $B \simeq H_{\rm ex}$, appropriate for a disk geometry, is reconsidered. The distinct thermodynamic roles of B and H, defined through the relation $H = 4\pi\partial f/\partial B$ (f is the appropriate free energy density), suggest that the two quantities should scale differently; this reflects the emergence of diamagnetic fluctuations. The distinction between the different fields is most apparent in a small "inner" scaling region near the zero field transition. In a realistic physical scenario, $H_{\rm ex}$ is the externally controlled variable; B and H then both acquire fluctuation contributions. Since estimates of the inverted XY scaling regime place it in the vicinity of the inner scaling region, it is therefore crucial to elucidate the differences between the different fields, in order to unravel the different critical phenomena.

The plan of the paper is as follows. Section II discusses the scaling form of the free energy, Eq. (2). Section III derives the magnetic equation of state, Eq. (5). The crossover field, H_0, between the inner and outer scaling regions, is identified in Eq. (8). Section IV derives the form of the superconducting phase transition line in the B-T, H-T, and $H_{\rm ex}$-T planes, given by Eqs. (9), (11), and (14), respectively. The H-T phase diagram is shown in Fig. 1. Section V is a brief interlude, which shows how the Abrikosov theory of the superconducting transition in a field is a special case of the more general scaling theory. The magnetization is discussed in Section VI, where we derive the relations between M and H [Eq. (25)] or M and $H_{\rm ex}$ [Eq. (26)], for $T = T_c$ or T_m. In Section VII we obtain the specific heat, Eq. (29). Section VIII postulates the dynamic scaling theory associated with the ohmic conductivity, Eq. (32). In Section IX we conclude, giving estimates for the size of the inner and inverted scaling regions.

Notation: In this paper, only the quantities with tildes involve factors related to sample geometry.

II. SCALING THEORY

We now discuss the basic thermodynamics of the scaling approach. At zero field, we adopt the usual scaling hypothesis, which states that any observed singular behavior involves the divergence of the correlation length, in terms of the relative temperature $t \equiv (T - T_c)/T_c$. The zero field correlation length $\xi(T)$ and specific heat $C(T)$ are then described by power laws, with the exponents ν_{xy} and α_{xy}, respectively [28].

To extend scaling to finite fields, we must know how B scales. We therefore introduce a definite assumption about the physics of a superconductor. A defining characteristic of a superconductor is its broken $U(1)$ or gauge symmetry, which is reflected in the symmetry of the superconducting order parameter. Gauge-invariance then implies the following identification for the gradient operator: $\boldsymbol{\nabla} \to \mathbf{D} = \boldsymbol{\nabla} + 2ie\mathbf{A}/\hbar c$. The basic scaling argument, which amounts to a dimensional analysis, states that the two terms appearing in $\mathbf{D}$ must have the same scaling dimension: $(\text{Length})^{-1}$. Similar dimensional arguments were proposed in Ref. [19], and later confirmed in Ref. [26]. The dimensionality of the magnetic field is then expressed as

$$B = |\boldsymbol{\nabla} \times \mathbf{A}| \sim (\text{Length})^{-2}. \tag{1}$$

The most general scaling hypothesis for the free energy density becomes [29]

$$f(B,T) = f_b(B,T) + f_k |t|^{2-\alpha_{xy}} \phi_\pm \left(\frac{B|t|^{-2\nu_{xy}}}{H_k} \right). \tag{2}$$

The function $f_b(B,T)$ represents the smooth background, while the second term in Eq. (2) encapsulates all the effects of 3D XY critical fluctuations. The scaling theory is universal in the sense that neither the exponents, ν_{xy} and α_{xy}, nor the functions ϕ_+ (ϕ_-), corresponding to $t > 0$ ($t < 0$), contain any sample dependence; they are the same for all superconductors which exhibit 3D XY scaling. The sample dependence rests only in the background term $f_b(B,T)$, the transition temperature T_c, and in the parameters f_k and H_k in the fluctuation term. Here, f_k has units of free energy density, making $\phi_\pm(x)$ dimensionless, while H_k has field units, making the scaling variable, $x = B|t|^{-2\nu_{xy}}/H_k$, dimensionless.

We now discuss several points concerning the free energy, Eq. (2):

(1) The arguments leading to Eq. (1) do not admit anomalous scaling dimensions, since the magnetic field does not fluctuate in the intermediate XY model class. On the other hand, fluctuations of the vector potential become important in the inverted XY critical region, and may lead to the appearance of an anomalous dimension: $B \sim (\text{Length})^{-2+\vartheta}$. Mounting experimental evidence seems to be consistent with the intermediate scaling presciption of $\vartheta = 0$. In the work which follows, the magnetic field is always treated within the intermediate scaling hypothesis. In particular, we emphasize that the inner scaling region, discussed below, also emerges

from Eq. (2), and is not related to inverted 3D XY behavior.

(2) The region of the B-T plane for which Eq. (2) provides the correct scaling description must either be determined experimentally, or by a more detailed theory. Empirically, the range of validity of Eq. (2) appears to extend far beyond the characteristic 3D XY temperature and field scales, t_1 and H_0 (discussed below), which are very small in many materials.

(3) Near $T = T_c$, the background term in f may be written approximately as follows:

$$f_b(B,T) \simeq f_0(T) + f_2(T)B^2, \tag{3}$$

where $f_0(T)$ and $f_2(T)$ are smooth functions, with no singular behavior. We shall see that $f_2(T)$ is related to the background magnetic susceptibility.

(4) The sample dependent quantities defined above, T_c, f_k, H_k, and $f_b(B,T)$, must all reflect the anistropy of the superconductor. The main effect of anisotropy on Eq. (2) is that H_k^{-1} becomes a tensor quantity. In other words, anisotropy causes H_k to depend on the direction of $\mathbf{B}$. To avoid such complications here, we may simplify the analysis by choosing our geometry carefully: we consider the simple, but experimentally relevant case that anisotropy, if present, is aligned with the principal axes of an assumed ellipsoidal sample. In addition, we require that any external magnetic field should be applied along a principal axis of the sample. In other geometries, the field $\mathbf{B}$ becomes nonuniform and/or misaligned with the external field $\mathbf{H}_{\rm ex}$.

(5) It is possible to relate the parameter f_k to another nonuniversal parameter ξ_0, which appears in the zero field critical correlation length $\xi(T) = \xi_0|t|^{-\nu_{xy}}$, by means of two-scale-factor universality [25,30]. The result can be written as $f_k = k_BT_c/\xi_0^3$, where for simplicity, we have absorbed a universal proportionality constant into the definitions of f_k and ξ_0. Without loss of generality, we may also introduce anisotropy into this relation as $f_k = \gamma k_BT_c/\xi_{ab0}^3$, where $\gamma = \xi_{ab0}/\xi_{c0}$ represents the ratio of the zero field 3D XY critical correlation lengths along different axes.

Several theoretical analyses of superconductors relate the renormalized anisotropy parameter, γ, to the anisotropy appearing in the bare Hamiltonian, by showing that the anisotropic problem may be treated as an isotropic one [16,31]. In this paper, we also assume that the isotropic and anisotropic problems should involve the same, universal scaling functions. However, we work strictly in terms of the measurable (renormalized) anisotropy factor. Note that unless ξ_{ab0} can be determined by independent means, it is not possible to extract γ directly from Eq. (2), without making further assumptions. (These are described in the following point.) We therefore retain the more general (f_k, H_k) notation here, noting that anisotropy is naturally absorbed into these parameters.

(6) The scaling ansatz of Eq. (2) is very general, since it arises from purely dimensional arguments. This form involves exactly two nonuniversal parameters [30], f_k and H_k, in addition to the temperature scale, T_c, and the background term, $f_b(B,T)$. It is not possible to reduce this number of sample dependent parameters without further information. Furthermore, we emphasize that the 3D XY model has not been solved exactly, and therefore cannot provide such information. However, it has been suggested that a relation exists between f_k and H_k [19,25], thus reducing the number of nonuniversal parameters by one.

The heuristic argument states that a characteristic field scale, $B_{\rm ch}(T)$, appears in the argument of the scaling functions $\phi_\pm(x)$, in the form $x = B/B_{\rm ch}(T)$. This field scale should be given precisely by [19,25] $B_{\rm ch}(T) = \Phi_0/\xi^2(T)$, where $\xi(T)$ is again the zero field correlation length. It follows that $H_k = \Phi_0/\xi_0^2$, from which we obtain the desired relation: $f_k/k_BT_c = (H_k/\Phi_0)^{3/2}$. For anisotropic superconductors, the relation becomes $f_k/\gamma k_BT_c = (H_k/\Phi_0)^{3/2}$, when $\mathbf{B}||\hat{\mathbf{c}}$, thereby providing a method for determining the anisotropy, γ, when the parameters f_k and H_k are determined experimentally.

The proposed relation between f_k and H_k can be tested through a scaling analysis of the fluctuation mangetization, as described in Section VI. We point out that the relation between certainly does not hold for the case of the mean field Abrikosov theory, described in Section V. However, for critical fluctuations, the question is still open. For generality here, we continue to treat f_k and H_k as independent parameters.

III. EQUATION OF STATE

To procede with the analysis of Eq. (2), the hyperscaling relation may be used: $2 - \alpha_{xy} = 3\nu_{xy}$ (for the 3D case). Eqs. (2) and (3) can be rewritten as

$$f(B,T) = f_0(T) + f_2(T)B^2 + f_k|t|^{3\nu_{xy}}\phi_\pm\left(\frac{B|t|^{-2\nu_{xy}}}{H_k}\right). \tag{4}$$

The magnetic equation of state is derived from the identity $H = 4\pi\,\partial f/\partial B$:

$$H = \Omega_T B + \left(\frac{4\pi f_k}{H_k}\right)|t|^{\nu_{xy}}\phi'_\pm\left(\frac{B|t|^{-2\nu_{xy}}}{H_k}\right), \tag{5}$$

where $\phi'_\pm(x) = \partial\phi_\pm/\partial x$. Eq. (5) relates B and H for a given temperature. The normal and fluctuation contributions appear in the first and second terms, respectively. $\Omega_T \equiv 8\pi f_2(T)$ is the inverse permeability of the background; experimentally, it is found that $\Omega_T \simeq 1$.

It is desirable to invert Eq. (5) to obtain $B(H,T)$. In general, this is not possible, because $\phi_\pm(x)$ are not yet

known theoretically. However, progress can be made in certain cases. When $B > 0$ and $T = T_c$, the fluctuation part of Eq. (5) must be smooth and independent of t. This leads to the following asymptotic behavior:

$$\lim_{x\to\infty} 4\pi\phi'_{\pm}(x) = (b_0 x)^{1/2}, \tag{6}$$

where, b_0 is a dimensionless, universal constant of the 3D XY theory. Thus when $T = T_c$, we have

$$\Omega_c B = H + 2H_0 - 2H_0^{1/2}\sqrt{H_0 + H}. \tag{7}$$

Here we have simplified the notation using $\Omega_c \equiv \Omega_{T_c}$.

In Eq. (7) we have defined a characteristic field

$$H_0 \equiv b_0 f_k^2 / 4\Omega_c H_k^3, \tag{8}$$

which represents the crossover between the inner and outer scaling regions, for $T = T_c$. This crossover field is very small for most superconductors, due to the small size of diamagnetic fluctuations above the transition. In the outer scaling region, $H \gg H_0$, we see that the usual approximation $H \simeq \Omega_c B$ is appropriate. However, in the inner region, $H \ll H_0$, diamagnetic fluctuations dominate the equation of state, leading to the behavior $B \propto H^{1/2}$, for $T = T_c$. If there exists an exact relation between f_k and H_k, as described in Sec. II, then Eq. (8) should reduce to $H_0 \propto \gamma^2 T_c^2$, demonstrating the strong anisotropy dependence of the inner scaling region becomes explicit.

IV. PHASE TRANSITION LINE

The free energy (4) must contain information concerning the phase transition into the superconducting state. Of crucial importance is the fact that the superconducting transition line $T_m(B)$ terminates at T_c on the $B = 0$ axis. The special point $(T = T_c, B = 0)$ is then multicritical [32]. For the case of a continuous transition considered in this paper, the transition always occurs at a particular, universal value of the scaling variable: $x = x_m$. The transition line is then given by

$$B(T) = (x_m H_k)|t|^{2\nu_{xy}}. \tag{9}$$

Combining Eqs. (5) and (9), we obtain the transition line in the H-T plane:

$$H(T) = \Omega_T x_m H_k |t|^{2\nu_{xy}} + \frac{4\pi f_k}{H_k}\phi'_-(x_m)|t|^{\nu_{xy}}. \tag{10}$$

Eq. (10) has two terms, one going as $|t|^{\nu_{xy}}$, and the other going as $|t|^{2\nu_{xy}}$. Two different types of nonuniversal parameters enter the $|t|^{2\nu_{xy}}$ term, including the background quantity Ω_T, which contains a possible temperature dependence. For simplicity, we will assume that Ω_T is nearly constant over the temperature range of interest: $\Omega_T \simeq \Omega_{T_c} \equiv \Omega_c$.

Eq. (10) then becomes

$$\frac{H(T)}{\Omega_c x_m H_k} = |t|^{2\nu_{xy}} + (t_1|t|)^{\nu_{xy}}. \tag{11}$$

We have introduced the relative temperature scale

$$t_1 \equiv (f_k b_1 / \Omega_c x_m H_k^2)^{1/\nu_{xy}}, \tag{12}$$

and the universal number $b_1 \equiv 4\pi\phi'_-(x_m)$. The expression $t_1 T_c$ represents the size of the inner scaling region along the transition line $T_m(H)$; it is analagous to H_0 on the H ax ase diag in Fig. 1.

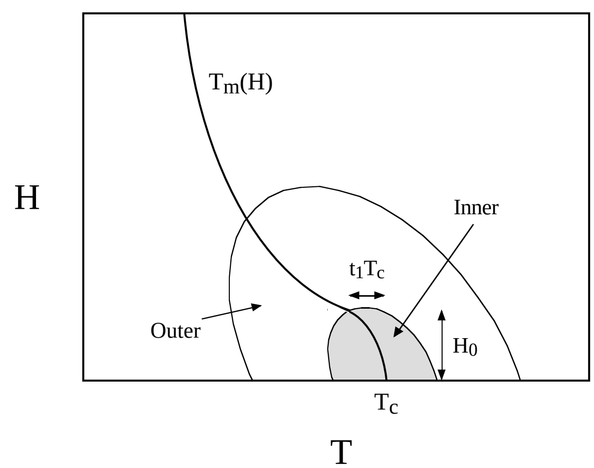

FIG. 1. Phase diagram in the H-T plane, showing the phase transition line $T_m(H)$ and the different scaling regions. In the shaded, "inner" scaling region, the usual approximation $B \simeq H$ breaks down. The dimensions of the inner region are of order $H_0 \times t_1 T_c$, where H_0 is defined in Eq. (8) and t_1 in Eq. (12). In the unshaded, "outer" region, the approximation $B \simeq H$ is accurate. Relative sizes of the two scaling regions are not drawn to scale; in many cases, the inner region is tiny compared to the outer region.

It is also of interest to study the phase transition line in the $H_{\rm ex}$-T plane, because of its experimental importance. We assume that $H_{\rm ex}$ is applied parallel to a principal axis of our sample, which is an ellipsoid [33]. A simple equation then relates the different fields [34]:

$$H_{\rm ex} = nB + (1-n)H, \tag{13}$$

where n is the demagnetizing coefficient, satisfying $0 \le n \le 1$. ($n \simeq 0.8 - 0.9$ for typical high-T_c single crystals, while $n \gtrsim 0.99$ for thin films.) Eqs. (9), (11), and (13) then give the transition line:

$$\frac{H_{\rm ex}(T)}{\tilde{\Omega}_c x_m H_k} = |t|^{2\nu_{xy}} + (\tilde{t}_1|t|)^{\nu_{xy}}, \tag{14}$$

where $(1-\tilde{\Omega}_T) \equiv (1-n)(1-\Omega_T)$, and $\tilde{t}_1^{\nu_{xy}} \equiv t_1^{\nu_{xy}}(1-n)\Omega_c/\tilde{\Omega}_c$. Note again that quantities with tildes differ from those without tildes only by geometric factors.

The limiting behaviors of Eqs. (11) and (14) are clear; we show only the result for $H_{\rm ex}$:

$$H_{\rm ex} \simeq \begin{cases} (\tilde{\Omega}_c x_m H_k \tilde{t}_1^{\nu_{xy}})|t|^{\nu_{xy}} & |t| \ll \tilde{t}_1, \\ (\tilde{\Omega}_c x_m H_k)|t|^{2\nu_{xy}} & |t| \gg \tilde{t}_1. \end{cases} \tag{15}$$

The large $|t|$ limit corresponds to the usual (outer region) description of the phase boundary [19]. New behavior is observed in the small inner region, $|t| \ll t_1$.

V. MEAN FIELD THEORY

As noted in the introduction, the superconducting transition in high-T_c materials should not be amenable to mean field description. However in low-T_c superconductors, the mean field description is often found to be accurate. It is therefore interesting to note that the Abrikosov theory [35] of the $B > 0$ mean field transition is consistent with the scaling theory presented so far. We now pause briefly to discuss this point.

Beginning with Eq. (2), progress cannot be made using hyperscaling, since this relation does not hold for the mean field transition. However, mean field exponents may be used, since they are known exactly: $\nu_{\rm mf} = 1/2$ and $\alpha_{\rm mf} = 0$. This gives

$$f_{\rm mf} = f_0(T) + \frac{B^2}{8\pi} + f_k|t_0|^2\phi_{\rm mf\pm}\left(\frac{B|t_0|^{-1}}{H_k}\right), \tag{16}$$

where $t_0 \equiv (T-T_{c0})/T_{c0}$, and T_{c0} refers to the temperature where the upper critical field vanishes: $B_{c2}(T = T_{c0}) = 0$. For simplicity, we have assumed that the normal state background has no magnetic effects ($\Omega_T = 1$), as is usual in the Abrikosov theory. The equation of state then becomes

$$H = B + \frac{4\pi f_k}{H_k}|t_0|\phi'_{\rm mf\pm}\left(\frac{B|t_0|^{-1}}{H_k}\right). \tag{17}$$

The mean field phase transition, $B_{c2}(T)$, occurs when the argument of $\phi_{\rm mf-}(x)$ has the value x_m, leading to the following transition line in the B-T plane:

$$B = (x_m H_k)|t_0|. \tag{18}$$

Using Eq. (17), the transition line in the H-T plane is given by

$$H = \left[x_m H_k + \frac{4\pi f_k}{H_k}\phi'_{\rm mf-}(x_m)\right]\frac{T_{c0}-T}{T_{c0}}. \tag{19}$$

The linearity of the transition line in $T_{c0} - T$ is consistent with the Abrikosov theory. We emphasize that this prediction, Eq. (19), has been obtained with no explicit knowledge of the Abrikosov ($B > 0$) solution. Instead, it is a general consequence of the scaling theory.

The scaling functions $\phi_{\rm mf\pm}(x)$ can now be explicitly computed. Recall the Abrikosov solution for the free energy, $f_{\rm mf}(B,T)$, near the upper critical field $H_{c2}(T)$ [35]:

$$f_{\rm mf} - f_0 = \begin{cases} \frac{B^2}{8\pi} - \frac{1}{8\pi}\frac{(H_{c2}-B)^2}{1+(2\kappa^2-1)\beta_A} & B \lesssim H_{c2}(T) \\ \frac{B^2}{8\pi} & B > H_{c2}(T) \end{cases}, \tag{20}$$

where

$$H_{c2}(T) \equiv \frac{\Phi_0}{2\pi\xi_{\rm BCS}^2}\frac{T_{c0}-T}{T_{c0}}. \tag{21}$$

Here, κ is the Ginzburg parameter, $\beta_A \simeq 1.16$ for the triangular vortex lattice, and $\xi_{\rm BCS} \sim \hbar v_F/k_B T_{c0}$ is the temperature independent coherence length.

The correspondence between Eqs. (16) and (20) becomes transparent by making the following identifications. The nonuniversal parameters can be taken as $H_k = \Phi_0/2\pi\xi_{\rm BCS}^2$ and $f_k = H_k^2[1+(2\kappa^2+1)\beta_A]^{-1}/8\pi$, where we have adopted the following normalization: $\phi_{\rm mf-}(0) = -1$. The quantities f_k and H_k contain all the sample dependence of the scaling description. Comparison with Eq. (16) now gives

$$\phi_{\rm mf-}(x) = \begin{cases} -(1-x)^2 & x \lesssim 1 \\ 0 & x > 1 \end{cases}, \tag{22}$$

$$\phi_{\rm mf+}(x) = 0. \tag{23}$$

Within the mean field approach, we can evaluate the universal quantities b_1 and x_m introduced earlier. We find that $b_1 = 0$, reflecting the lack of fluctuations in this case. (At temperatures above the transition, we simply have $B = H$.) Additionally, we find $x_m = 1$.

Finally, we point out that mean-field theory provides an example of a case where the nonuniversal parameters, f_k and H_k, cannot be related in a simple way, except by introducing an additional, nonuniversal parameter κ. However, H_k does take the form suggested by heuristic arguments in Sec. II. Note that the relation found here, $f_k \propto H_k^2$, differs from the one described in Sec. II, due to the breakdown of hyperscaling in mean field theory.

VI. MAGNETIZATION

In the remainder of this paper we consider only 3D XY critical fluctuations. As usual, the magnetization is defined by $M = (B-H)/4\pi$. Using Eq. (5), we obtain

$$M(B,T) = \frac{1-\Omega_T}{4\pi}B - \frac{f_k}{H_k^{3/2}}B^{1/2}\mathcal{M}_\pm\left(\frac{B|t|^{-2\nu_{xy}}}{H_k}\right), \tag{24}$$

where $\mathcal{M}_\pm(x) = x^{-1/2}\phi'_\pm(x)$ are universal scaling functions. The first term in Eq. (24) represents the normal state background, and vanishes for $\Omega_T = 1$. The second term is the critical fluctuation contribution.

As discussed in Sec. II, certain physical arguments may lead to a reduced number of nonuniversal parameters, by providing a relation $f_k H_k^{-3/2} \propto \gamma T_c$ ($\gamma = 1$ for the isotropic case), where the proportionality constant is not sample dependent. Eq. (24) offers a convenient experimental test of this prediction, since the expression $f_k H_k^{-3/2}$, can be directly infered from the scaling. To test the prediction, the quantities T_c and γ should be determined by independent methods [36].

Deriving equations for $M(H,T)$ or $M(H_{\rm ex},T)$ is not straightforward, because of the difficulty in inverting Eq. (5) to obtain $B(H,T)$. We therefore limit our derivation to the formulae relating M and H, or M and $H_{\rm ex}$, at the special temperatures T_c and T_m. Using the appropriate equations of state, we find the following results at $T = T_c$:

$$\begin{aligned} 4\pi\Omega_c M &= (1-\Omega_c)H + 2H_0 \\ &\quad - 2H_0^{1/2}\sqrt{H_0 + H}, \qquad (25) \\ 4\pi(1-n)\tilde{\Omega}_c M &= (1-\tilde{\Omega}_c)H_{\rm ex} + 2\tilde{H}_0 \\ &\quad - 2\tilde{H}_0^{1/2}\sqrt{\tilde{H}_0 + H_{\rm ex}}, \qquad (26) \end{aligned}$$

where $\tilde{H}_0 \equiv H_0(1-n)^2\Omega_c/\tilde{\Omega}_c$. The first term on the right hand side of both equations gives the normal background contribution, while the remaining terms represent the fluctuations. The asymptotic behavior of the fluctuation part of the magnetization, $M_{\rm fl}$, can be found. We show the results for $H_{\rm ex}$:

$$M_{\rm fl} \simeq \begin{cases} -\frac{H_{\rm ex}}{4\pi(1-n)\tilde{\Omega}_c} & H_{\rm ex} \ll \tilde{H}_0 \\ -\frac{\tilde{H}_0^{1/2} H_{\rm ex}^{1/2}}{2\pi(1-n)\tilde{\Omega}_c} & H_{\rm ex} \gg \tilde{H}_0 \end{cases}. \qquad (27)$$

In the low field limit, $M_{\rm fl}$ becomes asymptotically linear in H_{ex}, like the background. The fluctuation magnetization then dominates over the background by a factor proportional to $(1-n)^{-1}(1-\tilde{\Omega}_c)^{-1}$, which can be quite large for typical (flat) samples. Thus, in the low field ($H_{\rm ex} \ll \tilde{H}_0$), inner scaling region, the problem of background subtraction, which otherwise troubles experimental analyses, is ameliorated.

When $T = T_m$, we may still use Eqs. (25) and (26) by making the following replacements:

$$H_0 \to \left(\frac{b_1^2}{x_m b_0}\right) H_0 \quad \text{and} \quad \tilde{H}_0 \to \left(\frac{b_1^2}{x_m b_0}\right)\tilde{H}_0. \qquad (28)$$

VII. SPECIFIC HEAT

The free energy (4) may be used to compute the specific heat at constant B:

$$\begin{aligned} \frac{C(B,T)}{T} &= -f_0''(T) - f_2''(T)B^2 \\ &\quad + f_k T_c^{-2}|t|^{-\alpha_{xy}}\psi_\pm\left(\frac{B|t|^{-2\nu_{xy}}}{H_k}\right). \qquad (29) \end{aligned}$$

The dimensionless functions $\psi_\pm(x)$ depend on $\phi_\pm(x)$, and their first and second derivatives. The first two terms in Eq. (29) represent the background, while the last term is due to superconducting fluctuations.

We note the following points:

(1) Experimentally, it is difficult to isolate the fluctuation contributions in Eq. (29), due to (i) the smallness of fluctuations compared to the background, (ii) the weakness of the specific heat singularity [37] ($\alpha_{xy} \simeq -0.01$), (iii) rounding effects, which are often observed in experiments.

(2) It is possible to simplify Eq. (29) for the high-T_c materials, by using the empirical fact that $f_2''(T) \simeq 0$. The following scaling quantity may then be considered [5]:

$$\Delta C(B,T) \equiv C(B,T) - C(0,T), \qquad (30)$$

which contains no background dependence. However, if the zero field specific heat cusp becomes rounded near T_c, as is often the case for real samples, then the imperfect scaling, associated with the rounding, is transmitted to $\Delta C(B,T)$.

VIII. CONDUCTIVITY

Up to this point, we have developed our scaling analysis for thermodynamic quantities. All results have been derived from the expression for the free energy density, Eq. (4). However, the description of transport measurements, such as the conductivity, requires further information. Since the equations of motion governing the time evolution of the superconductor are not well understood, a dynamic scaling ansatz must be postulated.

Let us consider the ohmic conductivity σ, in order to avoid complications arising from current dependence [38]. For $B = 0$, when approaching T_c from above, σ diverges according to some power law. Fisher, Fisher, and Huse have given arguments leading to the following ansatz [19]:

$$\sigma_{\rm fl} \propto t^{-\nu_{xy}(z_{xy}-1)} \qquad B = 0, \qquad (31)$$

where $\sigma_{\rm fl}$ is the fluctuation part of σ, and z_{xy} is the exponent of the supposed 3D XY dynamic universality class.

Scaling at finite fields then proceeds in the usual way:

$$\sigma = S_b(B,T) + S_k|t|^{-\nu_{xy}(z_{xy}-1)}\Sigma_\pm\left(\frac{B|t|^{-2\nu_{xy}}}{H_k}\right). \qquad (32)$$

The first term, $S_b(B,T)$, represents the smooth background conductivity. The second term is the fluctuation contribution, where the sample-dependent parameter S_k has dimensions of conductivity. $\Sigma_+(x)$ $[\Sigma_-(x)]$ should be universal scaling functions, corresponding to $t > 0$ $[t < 0]$. The background conductivity term plays a relatively small role near the transition line, $T = T_m(B)$, due to the divergence of $\sigma_{\rm fl}$.

The functions $\Sigma_\pm(x)$ are not known theoretically, but we may deduce their asymptotic behavior. For $B > 0$ and $T \to T_c$, the conductivity should be finite, smooth, and independent of t, leading to

$$\lim_{x\to\infty} \Sigma_\pm(x) = s_0 x^{(1-z_{xy})/2}, \tag{33}$$

where s_0 is a universal number. Thus, at $T = T_c$,

$$\sigma_{\rm fl} = (S_k s_0)(B/H_k)^{(1-z_{xy})/2}. \tag{34}$$

A similar limit can also be taken for the conjugate fields; we show only the results for $H_{\rm ex}$:

$$\sigma_{\rm fl} = (S_k s_0)(H_k\tilde{\Omega})^{(z_{xy}-1)/2} \times \left[H_{\rm ex} + 2\tilde{H}_0 - 2\tilde{H}_0^{1/2}\sqrt{\tilde{H}_0 + H_{\rm ex}}\right]^{(1-z_{xy})/2}. \tag{35}$$

The asymptotic behavior of $\Sigma_-(x)$ as $T \to T_m(B > 0)$ is of particular interest when the transition is continuous. The multicritical description [32] involves a crossover from XY to glass fluctuations [5–8,19]. Although glass fluctuations dominate near $T_m(B > 0)$, the XY scaling fomula (32), is still appropriate. Approaching the transition, we find

$$\lim_{x\to x_m} \Sigma_-(x) \propto \begin{cases} (x - x_m)^{-\omega} & x > x_m \\ \infty & x < x_m \end{cases}, \tag{36}$$

where x_m is the same universal constant as in Section IV. The exponent ω is related to glass, not XY fluctuations. In Ref. [19] it has been argued that $\omega = \nu_g(z_g - 1)$, in analogy with Eq. (31), where ν_g and z_g are glass exponents.

IX. CONCLUSIONS

In this paper, we have presented the basic scaling theory of the intermediate 3D XY transition. The theory is quite general; it involves only the assumption that $B \sim (\text{Length})^{-2}$, which is deduced from minimal coupling. We stress that the theory should apply to all strongly type-II superconductors, including both high-T_c and low-T_c varieties. In particular, we have considered the case that the finite field transition $T_m(B)$ is continuous, although a similar analysis can be applied in the case of a first order melting transition [22].

The scaling results can be summarized by noting that the H-T or $H_{\rm ex}$-T superconducting phase diagrams involve two regimes, as shown in Fig. 1. In the outer region, scaling is the same for each of the fields B, H, and $H_{\rm ex}$, up to very small correction terms. To obtain results for the different fields, we consider the equations involving only B [for example, (4), (9), (24), (29), and (32)], then apply the following substitutions:

$$B \leftrightarrow \frac{H}{\Omega_T} \leftrightarrow \frac{H_{\rm ex}}{\tilde{\Omega}_T}. \tag{37}$$

In the inner region, scaling behaviors differ for B, H, and $H_{\rm ex}$. This is a nontrivial consequence of the conjugate nature of B and H, in the thermodynamic sense. Observation of the inner region should therefore be regarded as a more stringent test of 3D XY scaling. However, care must be taken to distinguish inner scaling behavior from inverted XY behavior. The dimensions of the inner region, in the H-T plane, are given by the characteristic field and temperature scales, H_0 and t_1T_c. In the $H_{\rm ex}$-T plane, these become $\tilde{H}_0$ and $\tilde{t}_1T_c$.

The high-T_c oxide materials are natural candidates for observing both the inner and outer 3D XY scaling behaviors, due to their strongly type-II character, and the prevalence of vortex fluctuations near the superconducting transition. However, recent experiments demonstrate behavior consistent only with the outer region. To understand this, it is helpful to obtain estimates for $\tilde{H}_0$, $\tilde{H}_{\rm eff}$, and $\tilde{t}_1$. We can make use of two experimental analyses. The first (I), by Hubbard *et al.* [12], involves a series of $YBa_2Cu_3O_{7-\delta}$ single crystals with varying δ. The second (II), by Moloni *et al.* [7], uses a similar series of thin films. Both cases involve samples with $T_c \simeq 77$ K. Using the appropriate demagnetizing factors ($n \simeq 0.84$ for sample I, and $n \simeq 0.992$ for sample II), we can make our estimates. We will refer to the geometry-independent properties of each sample (without tildes) as "intrinsic." These intrinsic quantities are assumed to be the same for both of the 77 K samples.

In Hubbard *et al.,* the fluctuation magnetization was plotted for $T \simeq T_c$, as a function of the field $H_{\rm ex}$. We can then use Eq. (27) to find $\tilde{H}_0$, with the assumption $\tilde{\Omega}_c \simeq 1$. This gives $\tilde{H}_0 \simeq 5 \times 10^{-7}$ Oe for sample I, corresponding to the intrinsic result $H_0 \simeq 2 \times 10^{-5}$ Oe. The estimate for sample II then becomes $\tilde{H}_0 \simeq 1 \times 10^{-9}$ Oe.

The relative temperature scale $\tilde{t}_1$ can be estimated by assuming that along the superconducting transition line, $H_{\rm ex} \simeq \tilde{H}_0$ when $|t| = \tilde{t}_1$. From Eq. (14), this gives $\tilde{t}_1 \simeq (\tilde{H}_0/2H^*)^{1/2\nu_{xy}}$, where $H^* \equiv \tilde{\Omega}_m x_m H_k$ is another characteristic field of experimental significance. For sample II, it was found that $H^* \simeq 19$ T. We can then estimate $\tilde{t}_1 \simeq 1\times10^{-9}$ for sample I and $\tilde{t}_1 \simeq 1\times10^{-11}$ for sample II, with the corresponding intrinsic result, $t_1 \simeq 2 \times 10^{-8}$. Note that the width of the inner scaling region of sample I becomes $\tilde{t}_1T_c \simeq 9 \times 10^{-8}$ K.

The estimates given above for H_0 and t_1 pertain to underdoped high-T_c cuprates. For the case of optimally doped cuprates, estimates for H_0 and t_1 should be somewhat smaller. Estimates for low-T_c superconductors should be even smaller. In each case, the smallness of the inner scaling region reflects the weak diagmagnetic response of fluctuations above T_m. Thus the inner region is probably experimentally inaccessible in many cases, due to sample inhomogeneities. However, strong anisotropy effects may improve the situation. As discussed in Sec. II, an assumed relation $(f_k/\gamma T_c)^2 \propto H_k^3$ leads to $H_0 \propto \gamma^2 T_c^2$, showing that H_0 (and t_1) may be greatly enhanced in very anisotropic samples.

It is also possible to estimate of the temperature width, $t_{\rm inv}$, of the inverted 3D XY scaling region. This may be compared to the size of the inner scaling region, t_1. Although neither the intermediate nor inverted 3D XY models has been solved exactly, the crossover may be estimated using the intermediate scaling theory: it is the point where our assumption of a uniform magnetic field breaks down. (See Sec. II.) Approaching the transition, the zero-field screening length [14,19], $\lambda = \lambda_0 |t|^{-\nu_{xy}/2}$, diverges more slowly than the 3D XY critical correlation length, $\xi = \xi_0 |t|^{-\nu_{xy}}$. We expect a crossover to occur when $\kappa = \lambda/\xi \simeq 1/\sqrt{2}$. The length scale ξ_0 of 3D XY fluctuations is not well known experimentally; however, a rough estimate can be obtained by using the bare Ginzburg-Landau parameter, which is easily obtained for samples such as I and II: $\kappa_0 = \lambda_0/\xi_0 \simeq$ 100-250. We finally obtain $t_{\rm inv} \simeq$ 2-40 $\times 10^{-8}$. In this rough estimate, the inner and inverted scaling regions are of comparable size, and must both be considered in order to correctly interpret fluctuations effects very near the 3D XY critical point.

In contrast to the elusive inner scaling region, the outer region, or behavior consistent with it, is readily observed over a wide temperature range: $|t| \lesssim 0.5$ [7], which is on the order of 10^9-10^{11} times $\tilde{t}_1$. The difference between the two XY temperature scales is striking. However, we note that there is no reason why they should be related. It remains an outstanding theoretical problem to provide estimates for the relevant temperature and field scales, from microscopic considerations.

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